

19a.) Homogeneous equation

$$\lambda^2 + 6\lambda + 8 = (\lambda + 2)(\lambda + 4) = 0$$

$$\Rightarrow \lambda^{(A)} = -2, \lambda^{(B)} = -4$$

(2P)

particular solution $y_p(x) = A e^{-x}$

$$= 0 \quad A e^{-x} - 6A e^{-x} + 8A e^{-x} = -3e^{-x} \Rightarrow A = -1$$

(4P)

general solution

$$y = C^{(A)} e^{-2x} + C^{(B)} e^{-4x} - e^{-x}$$

(2P)

b)

$$\lambda^2 + 3\lambda - 18 = (\lambda + 6)(\lambda - 3) = 0$$

$$\Rightarrow \lambda^{(A)} = -6, \lambda^{(B)} = 3$$

(2P)

$$y_p(x) = A e^{2x} \Rightarrow 4A e^{2x} + 6A e^{2x} - 18A e^{2x} = 18e^{2x}$$

$$\Rightarrow A = -\frac{9}{4}$$

(4P)

$$y = C^{(A)} e^{-6x} + C^{(B)} e^{3x} - \frac{9}{4} e^{2x}$$

(2P)

$$c) \lambda^2 + 9 = 0 \Rightarrow \lambda^{(A)} = 3i, \lambda^{(B)} = -3i$$

(2P)

$$y_p(x) = A \sin(2x) + B \cos(2x)$$

$$y_p' = 2A \cos(2x) - 2B \sin(2x)$$

$$y_p'' = -4A \sin(2x) - 4B \cos(2x)$$

$$= 0 \quad -4A \sin(2x) - 4B \cos(2x) + 9A \sin(2x) + 9B \cos(2x)$$

$$= \sin(2x) \quad (2P)$$

$$\Rightarrow 5A = 1, 5B = 0 \Rightarrow A = \frac{1}{5}, B = 0 \quad (2P)$$

$$y = C^{(A)} e^{3ix} + C^{(B)} e^{-3ix} + \frac{1}{5} \sin(2x) \quad (2P)$$

$$= D \sin(3x) + \frac{1}{5} \cos(3x) + \frac{1}{5} \sin(2x)$$

$$d) \lambda^2 + 7\lambda + 6 = (\lambda + 1)(\lambda + 6) = 0$$

$$\Rightarrow \lambda^{(A)} = -1, \lambda^{(B)} = -6$$

(2P)

$$y_p(x) = A \sin(2x) + B \cos(2x) \quad (y_p', y_p'' \text{ cf. c)})$$

$$= 0 \quad (-4A - 14B + 6A) \sin(2x) + (-4B + 14A + 6B) \cos(2x) = 3 \sin(2x) \quad (2P)$$

$$\Rightarrow 2A - 14B = 3 \Rightarrow A = \frac{3}{100} \quad B = -\frac{21}{100} \quad (2P)$$

$$\Rightarrow 14A + 2B = 0$$

$$\Rightarrow y = C^{(A)} e^{-x} + C^{(B)} e^{-6x} + \frac{3 \sin(2x) - 21 \cos(2x)}{100} \quad (2P)$$

$$e) \lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0$$

(2P)

$$\lambda^{(A)} = -2, \lambda^{(B)} = -3$$

$$y_p(x) = A + Bx + Cx^2$$

$$y_p' = B + 2Cx, \quad y_p'' = 2C$$

$$= 0 \quad 2C + 5B + 10Cx + 6A + 6Bx + 6Cx^2 = 4 - 6x^2 \quad (2P)$$

$$\Rightarrow 6A + 5B + 2C = 4$$

$$\Rightarrow$$

$$C = -1$$

$$10C + 6B = 0$$

$$B = \frac{5}{3}$$

$$6C = -6$$

$$A = -\frac{7}{18}$$

(2P)

$$y = C^{(A)} e^{-2x} + C^{(B)} e^{-3x} - \frac{7}{18} + \frac{5}{3}x - x^2$$

(2P)

$$2Ca) \quad \lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1) = 0$$

$$\Rightarrow \lambda^{(A)} = -2 \quad \lambda^{(B)} = -1$$

(2P)

$$y_p = A e^{-4x} \Rightarrow (-4)^2 A + 3(-4)A + 2A = 3$$

$$\Rightarrow A = \frac{1}{2}$$

(2P)

$$\Rightarrow y = C^{(A)} e^{-2x} + C^{(B)} e^{-x} + \frac{1}{2} e^{-4x}$$

(2P)

$$y' = -2C^{(A)} e^{-2x} + C^{(B)} e^{-x} - 2e^{-4x}$$

$$1 = y'(0) = C^{(A)} + C^{(B)} + \frac{1}{2}$$

(2P)

$$0 = y'(0) = -2C^{(A)} - C^{(B)} - 2$$

$$\Rightarrow C^{(A)} = -\frac{5}{2} \quad C^{(B)} = 3$$

(2P)

$$b) \quad \lambda^2 + 4\lambda + 5 = (\lambda + 2)^2 + 1 = 0$$

$$\Rightarrow \lambda^{(A)} = -2 + i, \quad \lambda^{(B)} = -2 - i$$

(2P)

$$y_p = A + Bx, \quad y_p' = B, \quad y_p'' = 0$$

(2P)

$$\Rightarrow 4B + 5A + 5Bx = 4 - x$$

$$\Rightarrow B = -\frac{1}{5} \quad A = \frac{24}{25}$$

(2P)

$$\Rightarrow y = C^{(A)} e^{(-2+i)x} + C^{(B)} e^{(-2-i)x} + \frac{24}{25} - \frac{1}{5}x \quad (2P)$$

$$y' = (-2+i)C^{(A)} e^{(-2+i)x} + (-2-i)C^{(B)} e^{(-2-i)x} - \frac{1}{5}$$

$$0 = y'(0) = C^{(A)} + C^{(B)} + \frac{24}{25}$$

(2P)

$$-1 = y'(0) = (-2+i)C^{(A)} + (-2-i)C^{(B)} - \frac{1}{5}$$

$$\Rightarrow C^{(A)} = -\frac{12}{25} + i\frac{34}{25}, \quad C^{(B)} = -\frac{12}{25} - i\frac{34}{25}$$

(2P)

$$c) \quad \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$$

$$\Rightarrow \lambda^{(A)} = 1 \quad \lambda^{(B)} = 2$$

(2P)

variation of parameters

$$y_p = A(x) e^x + B(x) e^{2x}$$

$$A(x) = \int \frac{e^{2x} e^{-x}}{1-2} dx = -e^x$$

(2P)

$$B(x) = \int \frac{e^{2x} e^{-2x}}{2-1} dx = x$$

$$\Rightarrow y_p = -e^{2x} + x e^{2x} \quad (\text{or } x e^{2x})$$

(2P)

$$y = C^{(A)} e^x + \underbrace{C^{(B)} e^{2x}}_{2C^{(B)} e^{2x}} - e^{2x} + x e^{2x}$$

(2P)

$$y' = C^{(A)} e^x + 2C^{(B)} e^{2x} + e^{2x} + 2x e^{2x}$$

$$0 = y(\omega) = C^{(A)} + 2C^{(B)}$$

(2P)

$$0 = y'(\omega) = C^{(A)} + 2C^{(B)} + 1$$

$$\Rightarrow C^{(A)} = -1$$

(2P)

$$C^{(B)} = -1$$

$$(C^{(B)} = 0)$$

$$2A.) \quad y = A \cos \phi \cdot \cos(\Omega x) - A \sin \phi \cdot \sin(\Omega x)$$

(4P)

$$y' = -\Omega A \cos \phi \sin(\Omega x) - \Omega A \sin \phi \cdot \cos(\Omega x)$$

$$y'' = -\Omega^2 A \cos \phi \cos(\Omega x) + \Omega^2 A \sin \phi \cdot \sin(\Omega x)$$

$$h \cos(\Omega x) = y'' + \delta y' + \omega^2 y$$

$$\begin{aligned} &= (-\Omega^2 A \cos \phi - \Omega \delta A \sin \phi + \omega^2 A \cos \phi) \cos(\Omega x) \\ &+ (\Omega^2 A \sin \phi - \Omega \delta A \cos \phi + \omega^2 A \sin \phi) \sin(\Omega x) \end{aligned}$$

(4P)

$$\Rightarrow h = -(\Omega^2 - \omega^2) A \cos \phi - \Omega \delta A \sin \phi$$

$$0 = (\Omega^2 - \omega^2) A \sin \phi - \Omega \delta A \cos \phi$$

(4P)

$$\Rightarrow h(\Omega^2 - \omega^2) = -(\Omega^2 - \omega^2)^2 A \cos \phi - (\Omega \delta)^2 A \cos \phi$$

$$h \Omega \delta = -(\Omega^2 - \omega^2)^2 A \sin \phi - (\Omega \delta)^2 A \sin \phi$$

$$\Rightarrow f_{\text{cond}} = \frac{f \Omega \delta}{\Omega^2 \omega^2}$$

(4P)

$$A^2 = \frac{h^2}{(\Omega^2 - \omega^2)^2 + (\Omega \delta)^2}$$

(4P)

if $\Omega = \omega$ and $\delta \rightarrow 0$ then

$A \rightarrow \infty$ ("resonance")

(4P)

$$6.) \quad \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$$

$$\Rightarrow \lambda = -1 \quad \text{repeated.}$$

$$y_p = \frac{1}{2} x^2 e^{-x} \text{ is a particular solution}$$

(4P)

by substitution or by using variation of parameters with

$$y = A(x) e^{-x} + B(x) \cdot x e^{-x}$$

$\rightarrow$ cf. textbook p 173ff).

$$y = C(x) e^{-x} + C(x) x e^{-x} + \frac{1}{2} x^2 e^{-x}$$

(4P)